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Reevaluating the debate on the effectiveness of alternative portfolio models: Out-of-sample analysis of Brazil's equity market, 2015-23

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Textos para Discussão

No. 19 – outubro 2024.

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Resumo:

Em todo o mundo, os mercados de ações chamam a atenção de investidores e pesquisadores financeiros, que compartilham um interesse comum na busca de estratégias de portfólio relativamente mais eficientes. Embora inúmeras novas técnicas de alocação tenham sido propostas, a literatura disponível ainda dá ênfase a sistemas analíticos mais tradicionais. Assim, nesta pesquisa, uma grande amostra, com muitas ações e longas séries de dados, é aplicada à análise comparativa de modelos de portfólio amplamente utilizados por meio de resultados baseados em dados fora-da-amostra. Os resultados obtidos, ao contrário de muitas avaliações críticas, destacam a superioridade das carteiras obtidas de modelos de alocação ótima sobre as estratégias baseadas em índice de mercado e pesos iguais.

Abstract:

Around the world, equity markets draw the attention of investors and financial researchers, who share a common interest in searching for relatively more efficient portfolio strategies. Although numerous new allocation techniques have been proposed, the available literature still gives emphasis to more traditional analytical systems. Accordingly, in this research, a large sample, with many stocks and long data series, is applied to the comparative analysis of widely used portfolio models through results based on out-of-sample data. The results obtained, contrary to many critical evaluations, highlight the superiority of portfolios derived from optimal allocation models over strategies based on market index and equal weights.

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Abstract
Around the world, equity markets draw the attention of investors and financial researchers, who share a common interest in searching for relatively more efficient portfolio strategies. Although numerous new allocation techniques have been proposed, the available literature still gives emphasis to more traditional analytical systems. Accordingly, in this research, a large sample, with many stocks and long data series, is applied to the comparative analysis of widely used portfolio models through results based on out-of-sample data. The results obtained, contrary to many critical evaluations, highlight the superiority of portfolios derived from optimal allocation models over strategies based on market index and equal weights.

Keywords: Equity markets; Asset allocation; Portfolio investment strategies; Portfolio performance; Mean-variance analysis.

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1. Introduction

Not long after the advent of mean-variance portfolio analysis (Markowitz, 1952), and despite the general recognition of its formal-theoretical refinement, it became clear that this model was subject to certain difficulties and inconsistencies—a finding based on evaluations of different types. In numerous articles on alternative portfolio strategies, it has been frequently mentioned that the application of the optimization model is not widespread among investment professionals, since the portfolios obtained by this method are often marked by weights considered "extreme" and "non-intuitive". In particular, a great number of assets are usually included with short positions – when the model is resolved without imposing non-negativity restrictions – or, when such constraints are introduced (long-only portfolios), solutions often include considerably large weights on assets with low liquidity (Michaud, 1989; Black & Litterman, 1992).

Another type of problem perceived with Markowitz's portfolio analysis is that, even with moderate revisions to the data, the weights in the portfolios are subject to exaggerated fluctuations. Moreover, in evaluations based on out-of-sample analyses, these solutions often do not perform favorably – in particular, when compared to straightforward portfolios with equal weights (De Miguel et al., 2009). Several authors have stated that, among the factors responsible for the shortcomings pointed out above, the problem of errors in the estimation of parameters stands out, especially in the case of expected returns – a situation that has been called "estimation risk". Deng et al. (2013) state that, in addition to errors in the estimation of means, risks, and correlations, the available data on returns are subject to high kurtosis and negative skewness.

A development that results from the perception of the importance of estimation errors is the line of research focused on risk, or "risk-based strategies" (De Carvalho et. al., 2012), since the absence of estimates of expected returns reduces the effects of estimation risk. In this context, the most common portfolio strategies can be prioritized according to the potential effects of estimation risk. In an initial position, portfolios with equal weights should be located given that this risk is not present. In a secondary place, should appear portfolios based exclusively on risk, a group that includes the minimum risk portfolio obtained from the mean-variance model. At the other extreme in this hierarchy should be included strategies that also use mean estimates and, in particular, portfolios that maximize the "ex-ante" Sharpe ratio.

The general objective of this paper is to pursue a comparative assessment of Markowitz's portfolio analysis from the perspective of the plentiful criticism available in the literature. To reach this overall aim, alternative strategies are evaluated using, on a monthly basis, out-of-sample data obtained for a large number of socks available in the Brazilian market. Each month, the portfolios were rebalanced by incorporating the latest data – a procedure that uses a fixed-sized "window" of data, which is periodically shifted. In this way, each month, ex-post returns that result from the solutions obtained in the previous period are available. In this evaluation, we consider both the unrestricted and restricted (long only) versions of the optimization model. We also impose an upper limit on the individual assets' weights in order to keep the results more closely related to the intuitive behavior of a typical investor.

In this study, the portfolio strategies that were used were: equal weights, sample-based mean-variance (global minimum variance), maximum ex-ante Sharpe ratio; global minimum variance with short-sale constraints; and maximum Sharpe ratio with short sale-constraints. Additionally, results for the major stock-market index in Brazil (Ibovespa) are used as a benchmark. On the other hand, a second type of comparison pursued is the statistical evaluation of the sample distributions of monthly returns in order to assess if the available data are consistent with the general hypothesis that the effects of estimation risk are present, especially in the case of solutions that maximize the Sharpe ratio. In this second type of comparison, the pattern observed in the equal weights' portfolios plays a central role.

Contrary to many of the critical evaluations available in the literature, but in accordance with the findings in a more recent study (Theron & van Vuuren, 2018), the overall Markowitz's analysis does not present an unfavorable performance in the data sample that was used.

2. Review of literature

One early reference that brought attention to the counter-intuitive nature and the problem of excessive variability in portfolios based on mean-variance analysis is Michaud (1989). This author also emphasizes that the data used in the optimized solutions were subjected to an acute problem of estimation error. As examples of problems of this kind, Black & Litterman (1992) point out that "when investors impose no constraints, the models almost always ordain large short positions in many assets" (p. 28). On the other hand, when non-negativity constraints are imposed, the optimizing solutions often contain "unreasonably" large weights in not-so-liquid assets.[1]

Zakamouline & Koekebakke (2009) affirm that, when one cannot conclude that the returns are normally distributed, solutions based on the maximum ex-ante Sharpe ratio can be "misleading" and "unsatisfactory". Additionally, Deng et. al. (2013) emphasize that, since the Sharpe ratio implicitly assumes that the returns are independently distributed normal random variables, this approach suffers from the problem of estimation errors given that this assumption is not valid in financial markets.

Among the articles investigating the efficiency of alternative portfolio allocation methods, one should mention Haugen and Baker (1991). These authors verify that indices based on market capitalization are not efficient in several situations, for example: when investors disagree about the risk and expected return; when short selling is restricted; when investment income is taxed; when investment alternatives are not included in the benchmark; and when foreign investors are in the domestic capital market. The authors conclude that, in these situations, there are alternatives to portfolios based on market capitalization that obtain the same expected return, but with less volatility. Alternatively, Grinold (1992), using an approach that was proposed in Gibbons et al. (1989), conducted tests on the possibility of outperforming the benchmark for five equity markets: German, American, Australian, British, and Japanese. The results indicated that for four of the five markets, the benchmarks (respectively: DAX, S&P 500, ALLORDS, FTA and TOPIX) were not efficient in the period analyzed.

Moreover, in terms of comparing alternative portfolio allocation models, a classic reference is DeMiguel et al. (2009), where several different models are examined and contrasted. These authors analyze out-of-sample data to access the potential effects of estimation risk of solutions based on mean-variance optimization. In particular, strategies with equal weights, minimum variances and maximum Sharpe ratios are included in the evaluation that was performed. Additionally, many authors consider that the problem of estimation errors is more pronounced in the case of expected returns than in moments of second order. This realization leads to the use of portfolio strategies that rely only on risk and diversification. De Carvalho et al. (2012) develop a very detailed empirical study to compare the main alternatives based on this approach, which also include minimum-variance solutions. In Braga (2015), a similar methodology is pursued of comparing

[1] Nevertheless, in the case of this latter problem, one way out is to introduce additional restrictions, with maximum values for the portfolio's weights – an approach that was in fact adopted in this research (Section 3).

portfolio strategies that require a smaller number of parameters in their solutions, since they are less exposed to estimation risk.

On the other hand, a recent article that is closer to the present study is Dolinar et al. (2017). These authors seek to evaluate the efficiency of market-capitalization-weighted indices (benchmarks) through the comparison with results obtained from traditional models: equal weights (or naïve), and the maximization of the Sharpe ratio. However, the authors did not consider the alternative of imposing non-negative solutions, as was contemplated in the present work. Theron & van Vuuren (2018) also present a comparative empirical analysis of portfolio strategies based on the mean-variance model and derive conclusions very similar to the ones in this study.

An earlier paper, that uses data for the Brazilian equity market, is Zanini & Figueiredo (2005). Although the approach followed in that text is different from the present research, the general objective is also to apply alternative portfolio models to data for the Brazilian stock market. In a contemporaneous work, Farias et al. (2006), using data for Brazilian equities, also present a comparative analysis for some portfolio selection models. More recently, Santos & Tessari (2012) assess out-of-sample performances of three portfolio allocation models, and contrast those with the Ibovespa benchmark. They also apply alternative estimators for the covariance matrix. In a different perspective, Naibert & Caldeira (2015), using data for Brazilian equities, examine minimum-variance models with alternative covariance matrix estimation methods. Also, Caldeira et al. (2017), investigate the eventual benefits of using high frequency data to construct optimal minimum-variance portfolios.

3. Mathematical overview

With the exception of the allocation strategy based on equal weights, the portfolio models in this paper are, from a mathematical perspective, examples of restricted optimization problems given that, in all of them, there is an equality constraint that is used to establish the main characteristic of a portfolio – represented by the vector x ($n\times1$). This constraint can be formulated as the linear function $s^T x = 1$, where $s^T = [1 \quad 1 \ldots 1]$. The general problem of optimization (minimum) with constraints can be represented by:

$$\min_{x} \quad f(x) \qquad (3.1)$$

$$\text{subject to} \quad h_i(x) = 0,\ i = 1, \ldots, m; \qquad (3.2)$$

$$g_j(x) \geq 0,\ j = 1, \ldots, p. \qquad (3.3)$$

In the case of portfolio optimization models that do not allow for short selling, besides the basic linear restriction of a portfolio, inequality constraints $x_j \geq 0$; $j = 1, \ldots, n$; are also present. In fact, in this research, additional restrictions were introduced in these versions of the optimization problem with the objective of imposing a maximum weight a given stock can have in the portfolio (25%) – which, from the perspective of an investor, does seem reasonable. Therefore, in these versions of portfolio optimization, the additional restrictions are: $0{,}25 - x_j \geq 0$; $j = 1, \ldots, n$.

The solutions of models that do not include inequality constraints have relatively simple analytic representations. On the other hand, models that include inequality constraints can only be solved through numerical methods – they do not have general analytical solutions. In all cases, however, the fundaments of these solutions are the same, and the general aspects are presented below (Simon & Blume, 1994; Gárciga-Otero, 2011).

The most common method to solve an optimization problem with constraints is based on the Lagrange function, which includes multipliers λ_i and μ_j:

$$L(x, \lambda, \mu) = f(\mathrm{x}) - \langle \lambda, \mathrm{h}(x) \rangle - \langle \mu, g(x) \rangle \qquad (3.4)$$

Proposition 1 (Karush-Khun-Tucker). Necessary conditions for a solution (x^*, λ^*, μ^*): Assuming that the partial derivatives of the functions f, h_i, g_j are well defined, then if x^* is a local solution (minimum point) of problem (3.1) – (3.3), then there are unique values x^*, λ^*, μ^* such that:

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0_n; \; h(x^*) = 0_m; \; g(x^*) \geq 0_p; \; \mu^* \geq 0_p; \; \text{and} \langle \mu^*, g(x^*) \rangle = 0 \quad (3.5)$$

Proposition 2. Sufficient conditions for a solution (x^*, λ^*, μ^*): Assuming that the first and second partial derivatives of the functions f, h_i, g_j are well defined, then one can construct the Hessian matrix $B = \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)$ which, being symmetric, represents a quadratic form $Q(x) = x^T Bx$. If $Q(x) > 0$, $\forall\, x \mid x \neq 0_n$ (positive definite), then a solution (x^*, λ^*, μ^*) that satisfies Proposition 1 is a local solution (minimum point) of problem (3.1) – (3.3).

From the perspective of a solution method and assuming $f(x)$ is quadratic and all restrictions are linear – which is the case of portfolio optimization problems –, Proposition 1 transforms the original (nonlinear) optimization problem into a linear problem. When inequality constraints are present, numerical solution methods must be used. One such approach is the so-called complementarity method (Murty, 1988; Miranda & Fackler, 2002).

It is well known that, in Markowitz's analysis, portfolios are associated with two variables, namely risk (or volatility) and expected return. In the R^2 space for these variables, the efficient frontier can be specified as the locus of portfolios with the highest expected return for a given level of risk (variance). For the general mean-variance problem, $f(x)$ is a positive definite quadratic function, and the sufficient conditions for a minimum (Proposition 2) apply. This optimization problem is (Vanini & Vignola, 2001):

General mean-variance portfolio optimization.

$$\min_{x} \quad \tfrac{1}{2}\, x^T V x \qquad (3.6)$$

$$\text{subject to} \quad x^T s = 1 \qquad (3.7)$$

$$x^T Er = r \qquad (3.8)$$

Matrix V in eq. (3.6) contains variances and covariances. Further, in eq. (3.8), Er represents a vector with the mean returns of the n assets, and r is a given value on the vertical axis. On the other hand, if restriction (3.8) is not included, then we have a global minimum-variance problem. When there are no inequality constraints – in particular, short sales are allowed –, the portfolio-optimization problem has a straightforward analytical solution.

Theorem 1. The solution of the global minimum-variance portfolio (3.6) – (3.7), without inequality restrictions, is $x^* = (1/a)\, V^{-1}$ s; $a = s^T V^{-1} s$.

For a proof, see Vanini & Vignola (2001) and Da Fonseca (2003).

By introducing in the above analysis the return of a risk-free asset, specified on the vertical axis, an optimum point on the frontier of efficient portfolios can be determined by a tangent line to the curve that contains the risk-free return. In this problem, restrictions (3.7) and (3.8) are altered to include a risk-free asset (r_f, with proportion invested x_0):

Optimum (tangent) portfolio on the efficient boundary.

$$\min_{x} \quad \tfrac{1}{2}\, x^T V x \qquad (3.6)$$

$$\text{subject to} \quad x^T s = 1 - x_0 \qquad (3.7a)$$

$$x^T Er = r - r_f\, x_0 \qquad (3.8a)$$

One aspect that deserves mention is that problem (3.6) – (3.8*a*) is equivalent to the maximization of the well-known Sharpe ratio.[2]

Theorem 2. The solution for the optimum (tangent) portfolio (3.6) – (3.8*a*), without inequality restrictions, is $x^* = V^{-1}\,(Er - r_f\, s)\,(b - a\, r_f)^{-1}$. In this solution, a is defined in Theorem 1, and $b = (Er)^T\, V^{-1} s$.

[2] Strictly speaking, for a given portfolio, the Sharpe ratio usually includes the (ex-post) return that was observed in a previous period.

For a proof, see Vanini & Vignola (2001) and Da Fonseca (2003).

4. Methodological elements and sample description

Generally, the stocks in the sample used in this research are the ones included in the benchmark index for the Brazilian equity market – the Ibovespa. The total number of equities in this benchmark is not fixed, since it usually changes with each revision of the index. In the three revisions that occurred in 2023, the total number of stocks were, respectively, 88, 85 and 86.

Initially 178 stocks were considered for inclusion in the sample. These stocks were available in a broader index for the Brazilian equity market – the IBrA-B3 – at the end of 2023. Then, from this initial set, only the stocks that were traded in the entire research period and, at the same time, were part of the Ibovespa in at least one edition during this period, were included in the sample that was actually used. The total number of equities that were incorporated in this final version was 72.

As previously stated, the main objective of this paper is to apply five alternative portfolio selection models to data available for Brazil's equity market from 2015 to 2023. In the context of the earlier researches mentioned in Section 2, the present work analyzes the following models: 1. Equal weights or naïve; 2. Sample-based mean-variance (global minimum variance); 3. Maximum Sharpe ratio; 4. Global minimum variance with short-sale constraints; and 5. Maximum Sharpe ratio with short sale-constraints.[3] Additionally, a comparison is also made with a benchmark for Brazil's equity market. In other to achieve this paper's goal, three basic procedures were implemented:

1. For the 72 stocks, estimates of mean returns, variances, covariances and standard deviations were obtained based on data available daily for the previous two years up to the last trading day of the month of reference.

2. Solutions are constructed for the alternative portfolio selection models using the estimates obtained in Stage 1.

3. For each portfolio selection model, out-of-sample returns were computed using data for the following month. That is, effective returns were obtained for the 72 stocks and this data were used to calculate the actual performance of the portfolio – from the perspective of an investor, these are the

[3] An important point is that the Sharpe ratio is not being considered here as an ex-post indicator of portfolio performance. Instead, the portfolio's (ex-ante) mean return is used in the traditional formula.

gains or losses that would occur by applying a given model.

Procedures 1, 2, and 3 were repeated for each month from January 2017 onwards through a “rolling window”, and this scheme provided out-of-sample results for 82 months.[4] In the models based on the Sharpe ratio, it was used the reference rate for Brazil’s Treasury bonds (Selic) for the riskless interest rate.

All computations were performed in the statistical software environment R, using several R financial functions available in the package fPortfolio that was developed by Rmetrics (Würtz et al., 2015).

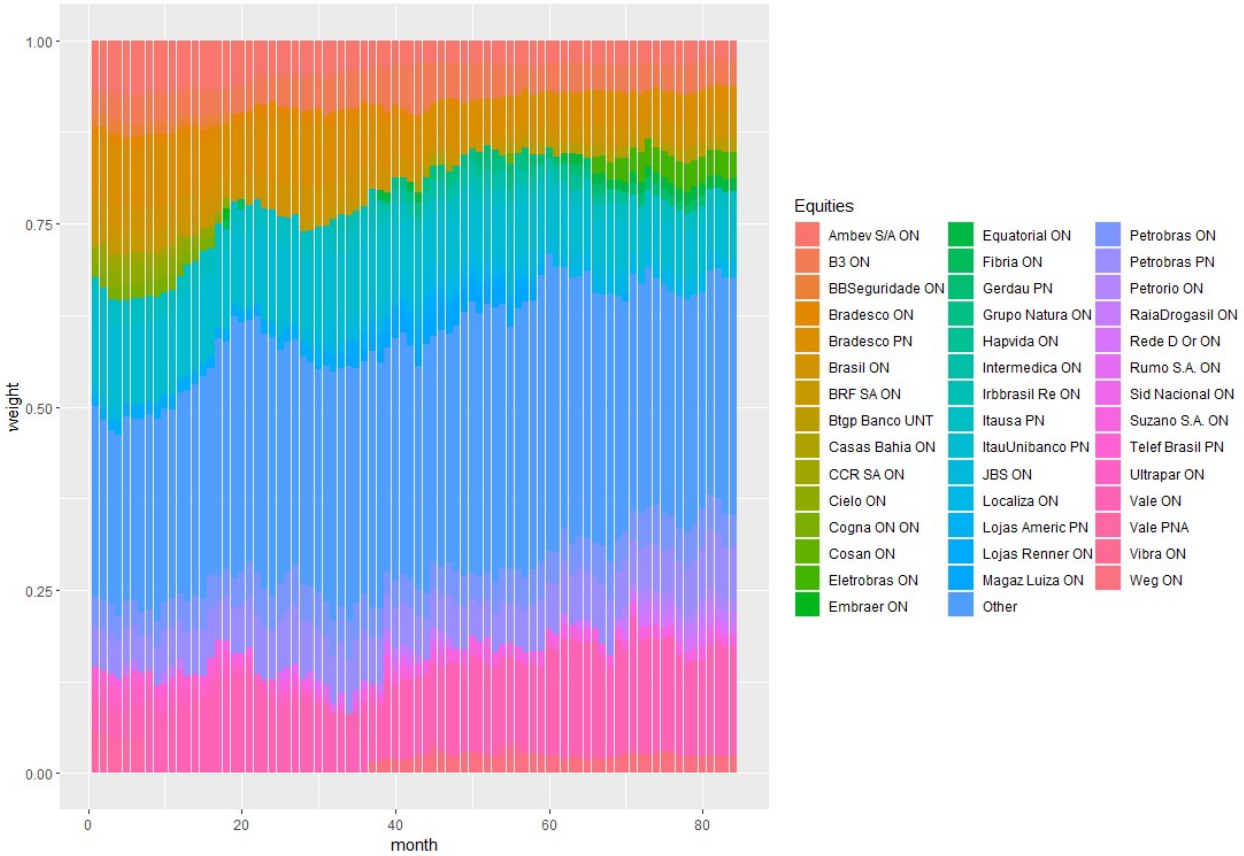
5. Analysis of the results

As mentioned in previous Sections, the Ibovespa was used as the benchmark for Brazil’s equity market. The use of weights based on broad indices like the Ibovespa is pehaps the most commom procedure that investors apply for portfolio allocation in equities – a solution that unfolds from investment analysis based on the CAPM model. Graph 1 gives a general perspective of the changes in this index in the 82 months for which, in the present study, out-of-sample results were obtained. As can be perceived in the Graph, there were no substantial changes in this benchmark during the research period.

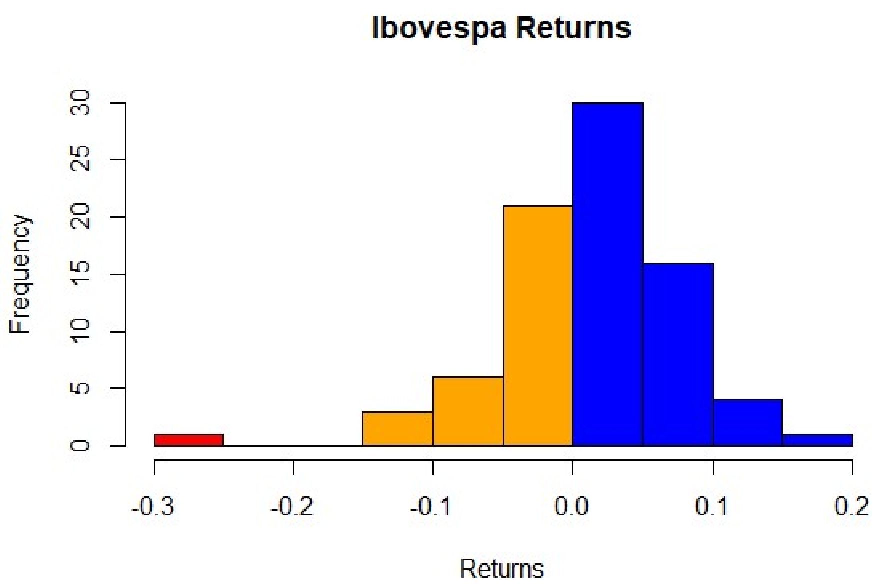
Graph 2 contains an histogram of monthly returns for the Ibovespa, and the dramatic effects of the covid epidemic are clearly illustrated on the left tail. For this reason, and in order to avoid greater discrepancies, the Table in Appendix A, with sample statistical indicators, includes only data up to 2019.

[4] It is important to emphasize that the procedure in Stage 3 creates a “real world” situation, in the sense that it simulates what would effectively happen to values invested in the portfolios with a one-month maturity.

GRAPH 1 – Ibovespa index: Equities with greater weights



GRAPH 2 – Ibovespa index: Monthly returns, 2015-23



From the perspective of someone who may invest in equity markets, the portfolios derived from the optimization models with inequality constraints – non-negative restrictions and maximum values for portfolio weights – certainly seem feasible and not too difficult to implement. This important aspect, from a practical standpoint, is illustrated in Graphs 3 (global minimum variance) and 4 (maximum Sharpe ratio). Further, these Graphs also reveal that, as intuition would suggest, the maximum Sharpe ratio portfolio is much more diversified and subjected to greater changes in its weights.

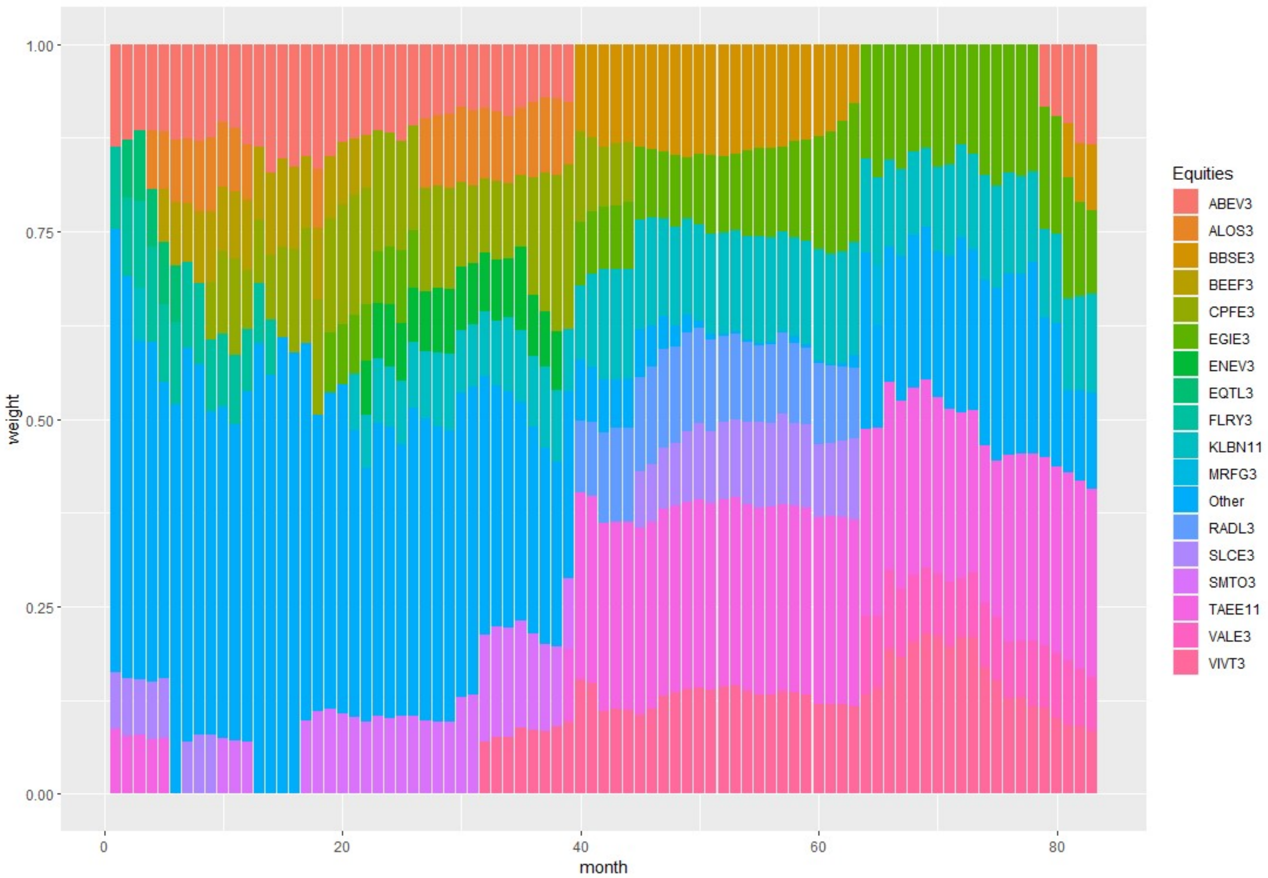
On the other hand, Graph 5 and Table 1 illustrate the results that should be considered the most relevant, especially given that they both are based on out-of-sample data that show the effectiveness of the alternative portfolios – that is, the gains and losses that would occur from a "real-world" investment strategy established at the beginning of each month.

In Graph 5, only returns are considered and, in relation to this variable, the huge disparity in the portfolio's performances is evident. It is especially noteworthy that one of the most common allocation strategies – perhaps the most common – based on passive investment in a benchmark portfolio, has shown considerably lower results than the others. In particular, it can be seen that the global minimum-variance portfolios were more successful in terms of cumulative return than the Ibovespa. In the case of models that maximize the Sharpe ratio, a more favorable performance in terms of cumulative returns can be expected. Also, the very favorable performance of the strategy based on equal weights should be highlighted.

Table 1, in turn, includes indicators for both average return and portfolio risk, as well as the ratio that combines them, suggested by Sharpe. Based on out-of-sample data, it can be seen that global minimum-variance portfolios have indeed met this target – they have the lowest risk. In addition, models that maximize the ex-ante Sharpe ratio, based on the mean return of the portfolio, effectively obtained the best results from ex-post data. By far the worst result in terms of the Sharpe ratio is that of the Ibovespa, which was outperformed by the naïve and global minimum-variance portfolios. Further, another aspect that deserves mention is that, considering the Sharpe-ratio results, the differences between optimization with and without restrictions were not very large. Still taking in the account the comparison between alternative portfolios, Graph 6 presents histograms for the best and worst portfolios in terms of the Sharpe ratio.

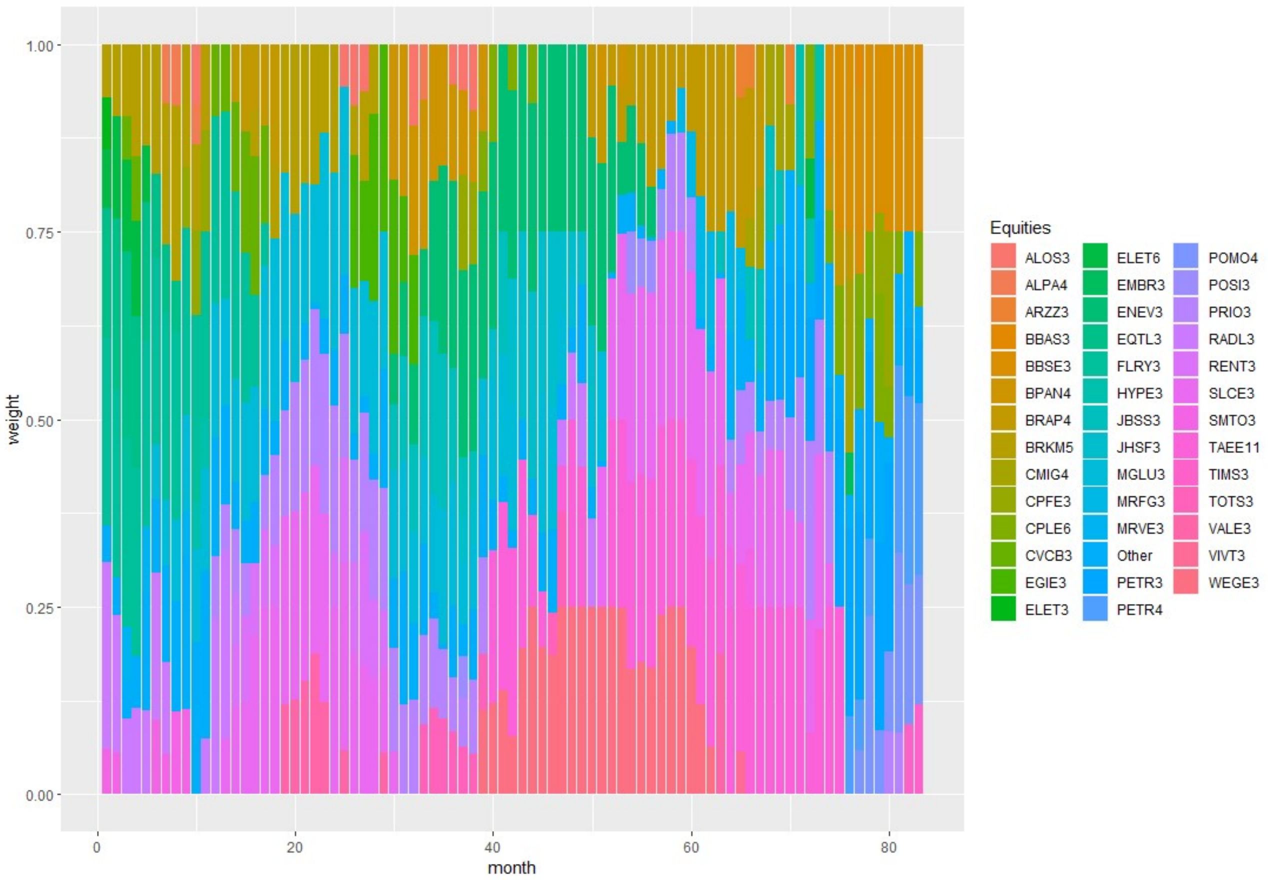
In summary, it seems adequate to conclude that what this research fundamentally shows is that the data available for the Brazilian stock market point to the superiority of portfolios based on Markowitz's mean-variance analysis, especially in the case of maximum Sharpe-ratio portfolios.

GRAPH 3 – Global minimum variance portfolios with inequality constraints
Equities with greater weights



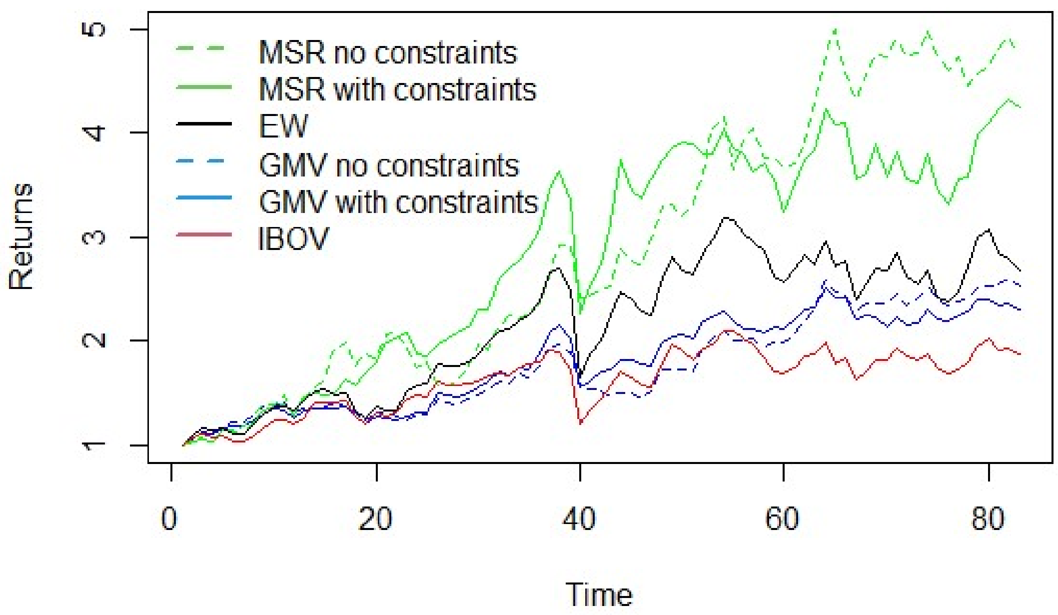
Note: Optimal solutions with non-negativity and maximum-weight inequalities.

GRAPH 4 – Maximum Sharpe ratio portfolios with inequality constraints
Equities with greater weights



Note: Optimal solutions with non-negativity and maximum-weight inequalities.

GRAPH 5 – Accumulated returns: Out-of-sample data



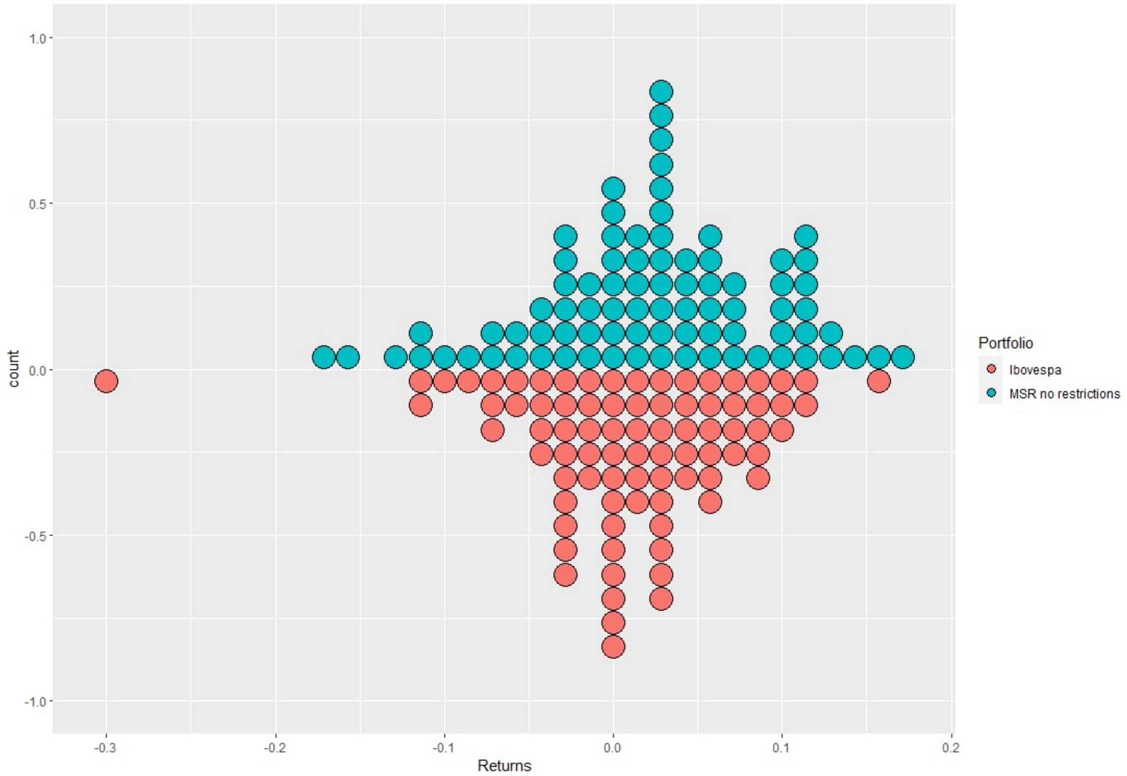
Notes: Data from out-of-sample monthly returns from Jan. 2017 to Oct. 2023.
MSR – Max. Sharpe ratio; EW – Equal weights; GMV – Global min. variance.

TABLE 1 - Out-of-sample portfolio performance (monthly returns)

	Ibovespa	Naïve	Global Minimum Variance		Maximum Sharpe Ratio	
			without restrictions	with restrictions	without restrictions	with restrictions
Average return	0.0099	0.0149	0.0126	0.0116	0.0216	0.0206
Acumulated return	1.8786	2.6706	2.5378	2.3131	4.7507	4.2455
Portfolio risk	0.0647	0.0733	0.0489	0.0500	0.0699	0.0737
Sharpe ratio	0.0415	0.1045	0.1100	0.0866	0.2057	0.1819

Note: The average risk-free monthly rate is 0.0072.

GRAPH 6 – Comparative histograms for portfolios with highest and lowest Sharpe ratio



Note: The lower histogram is the mirror image of Graph 2, with the difference that, in this case, each circle represents one month. The upper histogram has data for the maximum Sharpe ratio portfolio with no constraints.

6. Conclusions

Many evaluations of alternative portfolio models available in the literature include comparisons based on mean-variance analysis and broad market indices. In addition, the commonly used strategy of assigning equal weights to a relatively large number of assets is often included in these comparisons, given that this "naïve" strategy has shown favorable results in many applications. In general terms, these general lines for comparison of alternative portfolio were followed in the present study. More specifically, five models – four of them based on optimizing solutions – and a benchmark index were contrasted and compared using data for Brazil's equity market for a relatively long period of time.

In the sample that was used, only the stocks that were traded in the entire research period and, at the same time, were part of the Ibovespa in at least one edition during this period, were included – the total number of equities was 72. For each portfolio selection model, out-of-sample returns were computed using data for the following month after the period that was used in the estimation and solution. In this way, effective returns were generated, and they were used to calculate the actual performance of the alternative portfolios. This procedure was repeated for each month from January 2017 onwards through a "rolling window", and this scheme provided out-of-sample results for 82 months. In the models based on the Sharpe ratio, the reference rate for Brazil's Treasury bonds (Selic) was used for the riskless interest rate.

When only portfolio returns are considered (Graph 5), the huge disparity in the portfolio's performances is evident. It is especially noteworthy that one of the most common allocation strategies, based on passive investment in a benchmark portfolio, has shown considerably lower results than the others. On the other hand, it can be seen that the global minimum-variance portfolios were more successful in terms of cumulative return than the Ibovespa. In the case of models that maximize the Sharpe ratio, as might be anticipated, a more favorable performance in terms of cumulative returns was obtained. Also, the favorable performance of the strategy based on equal weights should be highlighted.

Considering indicators for both portfolio's return and risk, and especially the Sharpe ratio, it can be seen that, based on out-of-sample data, global minimum-variance portfolios have indeed met this target – they have the lowest risk. In addition, models that maximize the ex-ante Sharpe ratio, based on the mean return of the portfolio, effectively obtained the best results from ex-post data. By far the worst result in terms of the Sharpe ratio is that of the Ibovespa, which was outperformed by the naïve and global minimum-variance portfolios. Further, another aspect that deserves mention is that, considering the Sharpe-ratio results, the differences between optimization with and without restrictions were not very large.

It seems adequate to conclude, therefore, that what this research fundamentally shows is that available data for the Brazilian stock market point to the superiority of portfolios based on Markowitz's mean-variance analysis, especially in the case of maximum Sharpe-ratio portfolios.

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Appendix A: Sample statistical indicators

Daily returns (%)

Equities	Min.	Max.	Mean	Std.	Skew.	Kurt.
ABEV3	-8.649	8.17545	0.02546	1.38147	-0.0935	3.33286
ALOS3	-7.7601	9.69873	0.08828	1.76708	0.57509	2.65997
ALPA4	-11.676	13.815	0.14524	2.18915	0.1183	2.68745
ARZZ3	-11.778	8.29469	0.08133	2.27285	-0.1298	1.67649
B3SA3	-8.8165	9.63918	0.13258	2.07654	0.10257	1.21475
BBAS3	-23.789	13.4291	0.08189	2.80685	-0.477	7.999
BBDC3	-13.926	11.4074	0.0689	2.0192	-0.0395	2.84419
BBDC4	-14.056	12.2462	0.07196	2.06041	-0.0123	3.3952
BBSE3	-10.786	10.3523	0.03981	1.96205	-0.0402	2.57612
BEEF3	-8.2917	8.59033	0.00788	2.25774	0.15861	1.38153
BPAN4	-12.558	20.958	0.11887	3.16932	1.30138	7.08235
BRAP4	-28.085	15.3775	0.09719	3.12091	-0.4495	5.97215
BRFS3	-21.999	11.5806	-0.0442	2.2051	-0.5165	9.57721
BRKM5	-22.042	19.3927	0.05865	2.63072	-0.1403	10.0019
CCRO3	-15.415	10.5884	0.02762	2.31923	-0.2401	2.63373
CIEL3	-10.257	14.264	-0.0561	2.30788	0.35323	2.81565
CMIG4	-23.639	16.3848	0.02896	2.90892	-0.4499	7.77956
COGN3	-16.498	13.5223	-0.0222	2.93037	-0.0546	2.09887
CPFE3	-18.597	8.58726	0.06076	1.68999	-0.9601	14.2281
CPLE6	-13.477	8.98062	0.07762	2.40199	-0.2392	2.20887
CSAN3	-10.477	10.8449	0.07718	2.15076	-0.106	1.40418
CSNA3	-22.951	18.7511	0.05717	3.97169	0.19068	2.70897
CVCB3	-15.502	8.86188	0.09623	2.3182	-0.4173	3.71493
CYRE3	-17.749	11.0435	0.09593	2.27002	-0.2989	4.20235
DXCO3	-14.133	12.2486	0.07242	2.36826	0.1983	2.21913
ECOR3	-16.002	8.79688	0.06181	2.55369	-0.129	1.63166
EGIE3	-7.9922	6.37632	0.07577	1.46281	0.01155	2.12385
ELET3	-23.534	40.0759	0.14703	3.54509	1.25805	15.3772
ELET6	-18.587	27.8243	0.12425	3.1679	0.72171	6.85489
EMBR3	-16.783	20.2928	-0.0098	2.13957	-0.2407	13.0941
ENEV3	-44.183	28.7682	0.005	4.12125	-1.5359	24.8136
EQTL3	-6.4386	7.139	0.12637	1.48138	-0.0867	1.00619
EVEN3	-20.489	11.1005	0.09886	2.61566	-0.2088	3.69234
EZTC3	-13.777	10.2523	0.12543	2.32089	0.00975	1.48801
FLRY3	-7.0261	10.5318	0.12229	1.88372	0.20233	1.10403
GFSA3	-14.64	19.5131	-0.0551	3.18372	0.34078	2.57026
GGBR4	-12.81	14.9188	0.05133	3.03531	0.11576	1.54651
GOAU4	-20.955	16.2707	-0.0255	3.50688	-0.1712	3.34232
GOLL4	-22.444	40.7641	0.08786	4.49217	1.11252	10.3657
HYPE3	-15.383	19.18	0.07304	1.79186	0.4442	15.1094
ITSA4	-10.129	9.77579	0.08176	1.86249	-0.0672	2.15389
ITUB4	-12.836	10.3684	0.07263	1.90775	-0.0678	3.24672

JBSS3	-37.605	20.3245	0.07325	3.23634	-0.7193	18.3953
JHSF3	-19.807	25.4234	0.07358	3.22362	0.78116	5.22312
KLBN11	-6.6559	6.31171	0.04616	1.71487	0.0944	0.70082
LREN3	-8.0651	9.34271	0.12862	1.97652	0.17355	0.83232
MGLU3	-17.751	31.6912	0.30684	3.99687	0.95732	8.30178
MRFG3	-10.583	17.2358	0.04276	2.63692	0.6363	3.53496
MRVE3	-8.6136	12.1949	0.11649	2.17324	0.18969	1.33659
MULT3	-13.293	6.69031	0.0666	1.80422	-0.1001	2.80589
NTCO3	-12.506	13.6711	0.06895	2.46927	0.35952	2.73693
PETR3	-16.154	14.9662	0.0712	3.1187	-0.021	3.01135
PETR4	-17.149	15.0858	0.06752	3.19845	-0.1391	3.33562
POMO4	-10.178	13.6361	0.01981	2.60887	0.29573	1.71455
POSI3	-16.252	31.3483	0.12715	3.33236	0.97549	9.45775
PRIO3	-26.358	60.7989	0.15916	4.5917	2.47063	30.8367
QUAL3	-34.77	31.2153	0.0705	2.87051	-0.763	30.3462
RADL3	-6.636	8.84554	0.1213	1.83815	0.24332	0.96868
RENT3	-7.3285	8.02942	0.12929	2.23208	0.04611	0.58522
SANB11	-11.819	8.45574	0.12059	2.121	-0.1107	1.77555
SBSP3	-12.386	10.4021	0.10088	2.25769	-0.2811	2.41951
SLCE3	-9.323	9.30265	0.11032	2.26761	0.08426	1.39214
SMTO3	-9.6412	9.68997	0.05936	1.79218	0.09813	2.33309
TAEE11	-8.674	8.37174	0.08679	1.5851	-0.1759	1.66165
TIMS3	-8.9029	10.7098	0.02354	2.01964	0.01881	1.54117
TOTS3	-7.4662	8.27476	0.05799	2.05971	0.0702	1.052
UGPA3	-10.717	8.32473	0.00931	1.86238	-0.1859	2.51572
USIM5	-17.598	30.0892	0.04546	3.94261	0.47645	4.91697
VALE3	-28.135	13.7685	0.07905	3.10285	-0.5497	6.75426
VIVT3	-10.831	8.72521	0.04321	1.8545	-0.0148	2.28373
WEGE3	-9.5453	6.85752	0.09657	1.76426	-0.127	1.06274
YDUQ3	-16.434	21.2984	0.06069	3.09667	0.02274	3.37932